Concentration of Multilinear Functions of the Ising Model

And Applications to Network Data

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Ising Models

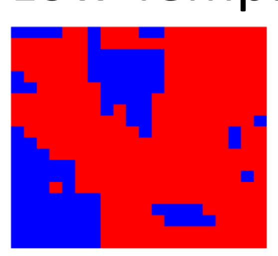
- **Graphical models** – Used to model high-dim data. E.g. pixels, genomes, social networks.
- **Ising Models:** Canonical example; captures binary MRFs with pairwise interactions.
- Distribution over $\{-1,+1\}^n$.

$$P_\theta(x_1, x_2, \ldots, x_n) \propto \exp\left(\sum_u \theta_u x_u + \sum_{u \neq v} \theta_{uv} x_u x_v\right)$$

Local field — Pairwise Interaction

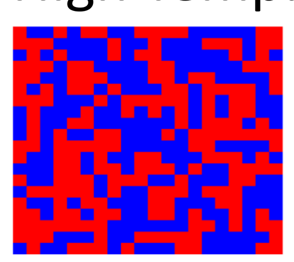
- Applications:
  - Physics
  - Computer Vision
  - Social networks
  - Neuroscience

Low Temp.



Large $\theta_{uv} \leftrightarrow$ Strong interactions

High Temp.



Small $\theta_{uv} \leftrightarrow$ Weak interactions

Contributions

THEORY:

- **Tight concentration bounds for Polynomial functions of Ising model**
- Natural statistics – useful for hypothesis testing, learning.
- Handle complex dependencies between nodes.
- Improve previously known log-Sobolev bounds.
- Contribution to probability, statistical physics.

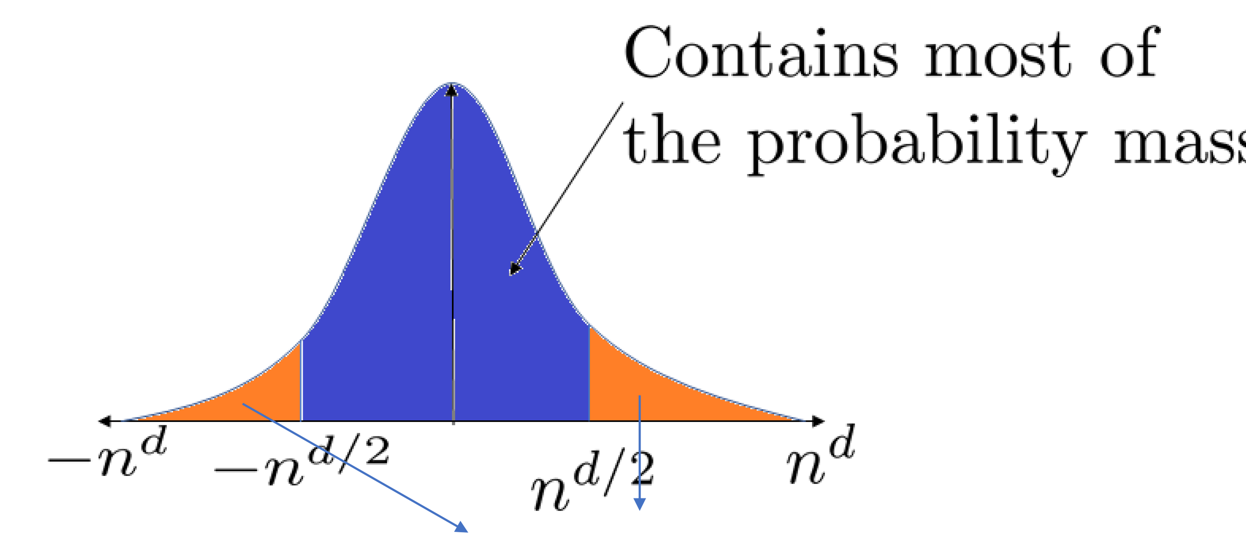
APPLICATIONS:

- **Test existence of strong network effects**
  - In synthetic data.
  - In **Last.fm** dataset.

What if all variables are I.I.D.?

If each $X_i \sim Rad\left(\frac{1}{2}\right)$ i.i.d., then know tight concentration for d-linear function $f(.)$

Radius: $O_d(n^{d/2})$ **Tail (r):** $2\exp\left(-\frac{r^{2/d}}{O_d(n)}\right)$



$$\leq 2\exp\left(-\frac{r^{2/d}}{O_d(n)}\right)$$

Our Result

Known that no non-trivial concentration holds in low-temperature.

We show **High Temp. ≈ Independent case.** (weaker assumption of ***Dobrushin's uniqueness*** suffices)

Previous Best (degree-d function):
Radius of conc. = $O_d(n^{d-0.5})$
Our bound: **Radius of conc. =** $\tilde{O}_d(n^{d/2})$

Also achieve tight decay rate of tail:

$$Pr[|f(X) - E[f(X)]| > r] \leq 2\exp\left(-\frac{r^{2/d}}{\tilde{O}_d(n)}\right)$$

Proof Idea: Martingale Theory

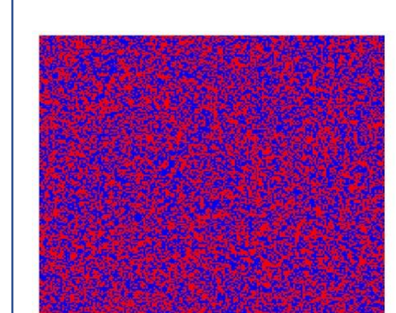
- Many techniques known when independence exists.
- **Main challenge**: Dependent random variables.
- Known Log-Sobolev techniques suboptimal.

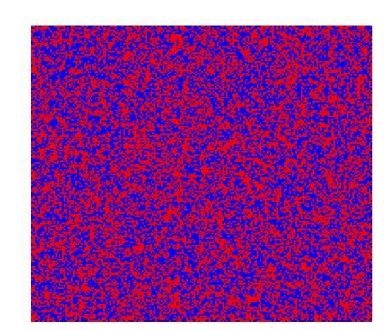
- **Our approach:** Martingale theory.
- Use **Glauber dynamics** together with concept of **Doob martingale.**
- **Freedman's inequality:** Builds on Azuma's inequality, stopping time.
- **Inductive Structure:** (d-1)-degree concentration crucial for d-degree concentration.

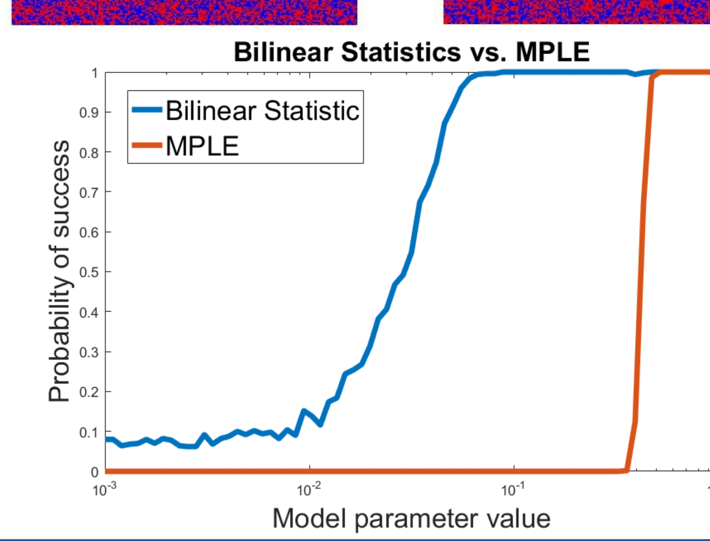
Application: Testing Strong Network Effects

From *One* Sample

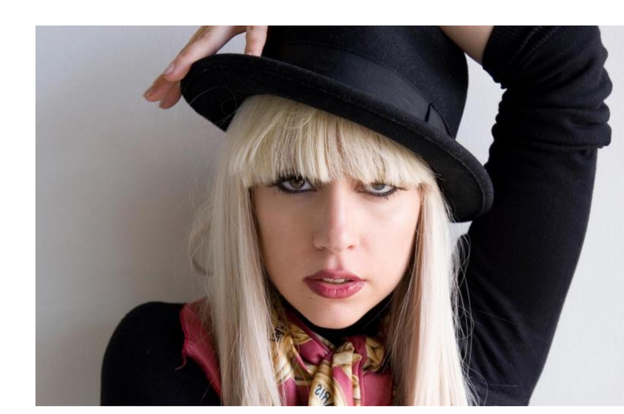
Which of the two images is from a true Ising model?



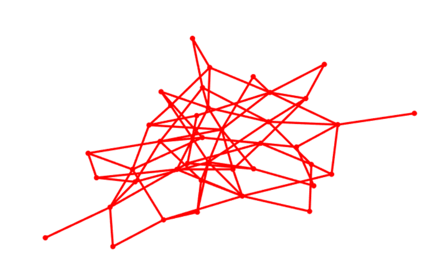




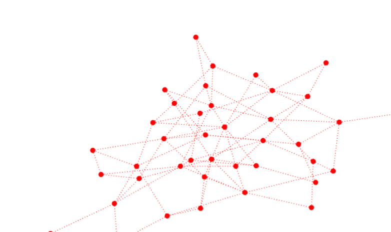
Last.fm Data



vs



vs



Strong network effects — Weak network effects