

Bounds on the Expectation of the Maximum of Samples from a Gaussian

Gautam Kamath

In this document, we will provide bounds on the expected maximum of n samples from Gaussian distribution. Let $Y = \max_{1 \leq i \leq n} X_i$, where $X_i \sim \mathcal{N}(0, \sigma^2)$ are i.i.d. random variables. We will show that

$$0.23\sigma\sqrt{\log n} \leq \mathbf{E}[Y] \leq \sqrt{2}\sigma\sqrt{\log n}$$

First, we show $\mathbf{E}[Y] \leq \sigma\sqrt{2}\sqrt{\log n}$. This result and method are folklore, but we include them here for completeness.

$$\begin{aligned} \exp(t\mathbf{E}[Y]) &\leq \mathbf{E}[\exp(tY)] \\ &= \mathbf{E}[\max \exp(tX_i)] \\ &\leq \sum_{i=1}^n \mathbf{E}[\exp(tX_i)] \\ &= n \exp(t^2\sigma^2/2) \end{aligned}$$

The first inequality is Jensen's inequality, the second is the union bound, and the final equality follows from the definition of the moment generating function.

Taking the logarithm of both sides of this inequality, we get

$$\mathbf{E}[Y] \leq \frac{\log n}{t} + \frac{t\sigma^2}{2}$$

This can be minimized by setting $t = \frac{\sqrt{2\log n}}{\sigma}$, which gives us the desired result

$$\mathbf{E}[Y] \leq \sigma\sqrt{2}\sqrt{\log n}.$$

Next, we show the more difficult direction, the lower bound. The desired inequality can be verified for $1 \leq n \leq 5$ by inspecting the explicit equations for $E[Y]$. Therefore, we will focus on $n \geq 6$.

Note that we have the following crude bound:

$$\begin{aligned} \mathbf{E}[Y] &\geq P(|\{i : X_i \geq 0\}| \geq \lceil n/3 \rceil) \cdot \mathbf{E} \left[\max_{1 \leq i \leq \lceil n/3 \rceil} |X_i| \right] + P(|\{i : X_i \geq 0\}| < \lceil n/3 \rceil) \cdot \mathbf{E}[|X_i|] \\ &\geq \frac{3}{4} \cdot \mathbf{E} \left[\max_{1 \leq i \leq \lceil n/3 \rceil} |X_i| \right] - \frac{\sigma}{4} \cdot \sqrt{\frac{2}{\pi}}. \end{aligned}$$

The second inequality uses the expected value of the half-normal distribution.

It remains to lower bound $\mathbf{E}[\max_{1 \leq i \leq k} |X_i|]$. We will show that

$$\Pr(|X_i| \geq \sigma\sqrt{\log n}) \geq \frac{3}{n}.$$

This will imply the following lower bound:

$$\begin{aligned}
\mathbf{E} \left[\max_{1 \leq i \leq \lceil n/3 \rceil} |X_i| \right] &\geq \sigma \sqrt{\log n} \cdot P(\exists i : |X_i| \geq \sigma \sqrt{\log n}) \\
&\geq \sigma \sqrt{\log n} \cdot \left(1 - \left(1 - \frac{3}{n} \right)^{\lceil n/3 \rceil} \right) \\
&\geq \left(1 - \frac{1}{e} \right) \sigma \sqrt{\log n}
\end{aligned}$$

We compute the CDF of $|X_i|$ at the point $\sigma \sqrt{\log n}$.

$$\begin{aligned}
P(|X_i| \geq \sigma \sqrt{\log n}) &= 1 - \operatorname{erf} \left(\frac{\sqrt{\log n}}{\sqrt{2}} \right) \\
&\geq 1 - \sqrt{1 - \exp \left(-\frac{2}{\pi} \log n \right)} \\
&= 1 - \sqrt{1 - n^{-\frac{2}{\pi}}},
\end{aligned}$$

where the first equality is based on the CDF of the half-normal distribution and the inequality is from the bound on the error function, $\operatorname{erf}(x) \leq \sqrt{1 - \exp(-\frac{4}{\pi}x^2)}$. We require this value to be at least $\frac{3}{n}$:

$$\begin{aligned}
1 - \sqrt{1 - n^{-\frac{2}{\pi}}} &\geq \frac{3}{n} && \Leftrightarrow \\
1 - \frac{3}{n} &\geq \sqrt{1 - n^{-\frac{2}{\pi}}} && \Leftrightarrow \\
1 - \frac{3}{n} + \frac{9}{n^2} &\geq 1 - \frac{1}{n^{\frac{2}{\pi}}} && \Leftrightarrow \\
n^{2-\frac{2}{\pi}} &\geq 3n - 9 && \Leftrightarrow \\
\left(2 - \frac{2}{\pi} \right) \log n &\geq \log(3n - 9) && \Leftrightarrow \\
\left(2 - \frac{2}{\pi} \right) \frac{\log n}{\log(3n - 9)} &\geq 1
\end{aligned}$$

This inequality holds for all $n \geq 6$, as desired.

Putting these inequalities together, we have

$$\mathbf{E}[Y] \geq \frac{3}{4} \left(1 - \frac{1}{e} \right) \sigma \sqrt{\log n} - \frac{\sigma}{4} \sqrt{\frac{2}{\pi}} \geq 0.23 \sigma \sqrt{\log n},$$

where the second inequality holds for any $n \geq 2$.

Other proofs. Proofs of qualitatively similar lower bounds also appear in [vH14] and [OP15].

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References

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- [vH14] Ramon van Handel. Probability in high dimension. Technical report, Princeton University, 2014.